Subregular Constraints

Jeffrey Heinz

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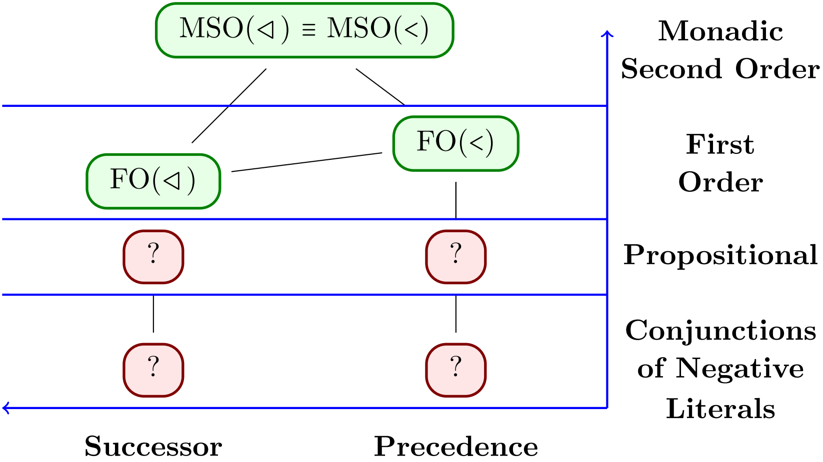
Part I

Overview of Today

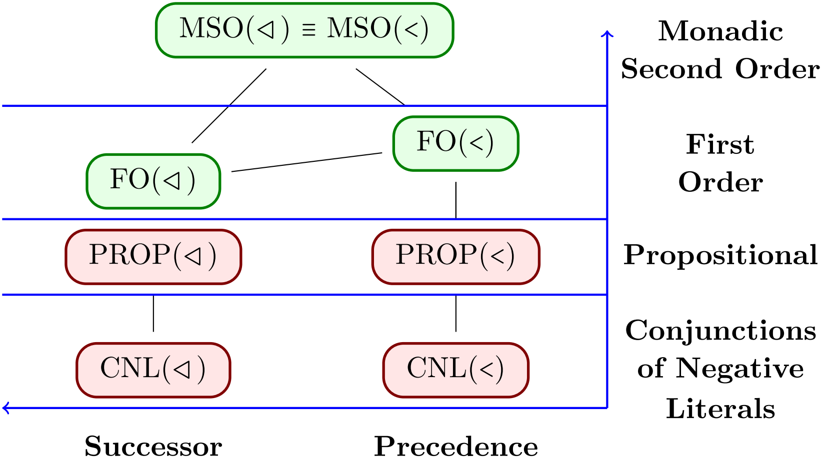
Going Below First Order

1. We have defined constraints and transformations with MSO logic and various model signatures.
2. We have also studied constraints and transformations using only FO logic and various model signatures.
3. Today we go below First Order…

Hierarchies of Constraints



Hierarchies of Constraints



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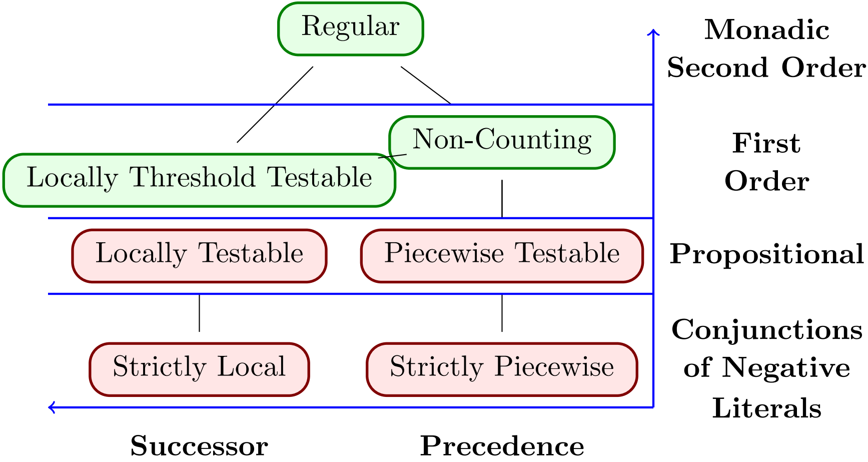
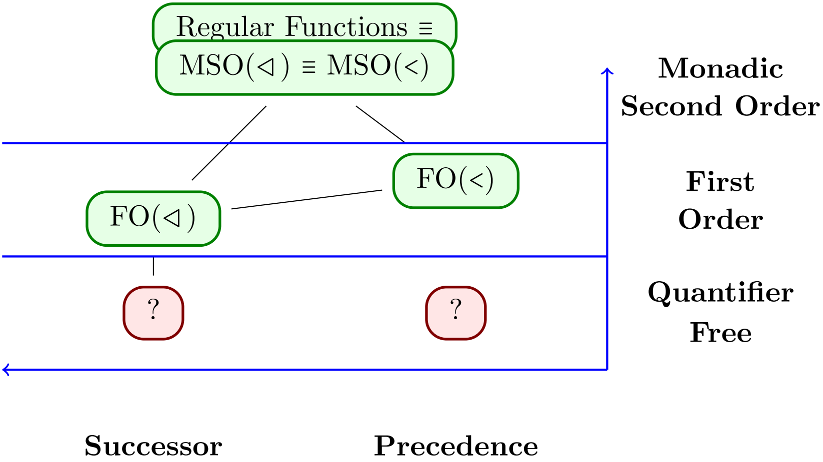


Hierarchies of Transformations



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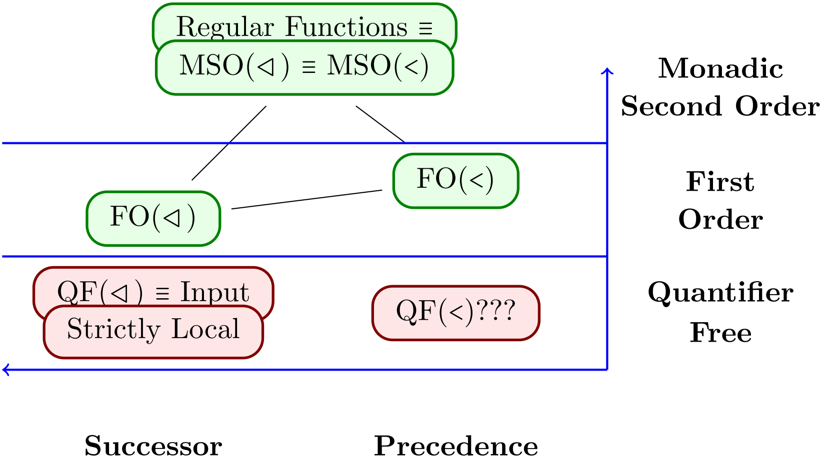

Today

1. Filling out Hierarchies of Constraints
   - PROP($\triangleleft$)
   - CNL($\triangleleft$)
   - PROP($<$)
   - CNL($<$)
   - …more representations, more logics?
2. Filling out Hierarchies of Transformations
   - QF($\triangleleft$) transformations are Input Strictly Local transformations.
   - The problem with characterizing long-distance transformations as QF($<$)

Abstract characterizations...

1. are independent of logical formulas, grammars, and automata
2. provide laws of *inference* for learning
3. provide ways to show certain stringsets do NOT belong to the class.

(Rogers and Pullum 2011, Rogers et al. 2013)

Part II

Hierarchies of Constraints

Propositional Logic

Syntax

1. Base cases: There is a set of atomic propositions $\{P, Q, R, S, \ldots\}$, each element of which is a sentence of propositional logic.
2. Inductive case: If φ, ψ are sentences of propositional logic, then so are:
   - $\neg\varphi$ (negation)
   - $(\varphi \wedge \psi)$ (conjunction)
   - $(\varphi \vee \psi)$ (disjunction)
   - $(\varphi \Rightarrow \psi)$ (implication)
   - $(\varphi \Leftrightarrow \psi)$ (biconditional)

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Semantics

1. The atomic propositions are assigned values typically from a truth table.
2. The inductive cases are determined compositionally in the usual way.

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- The logical atoms (also called *literals*) are *connected structures* in relational models. We call them *factors*.

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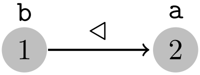
Interpretation of atoms

$\mathcal{M}_w \models \mathcal{S}$ if and only if $\mathcal{S} \sqsubseteq \mathcal{M}_w$.

(Rogers and Lambert 2019, Chandlee et al 2019)

Examples

In the successor model, the model of **ba**



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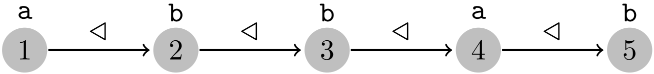
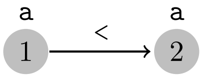


Examples

In the precedence model, the model of **aa**



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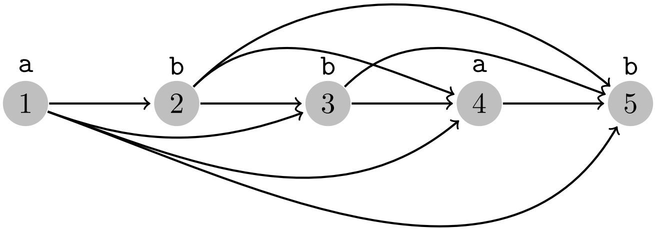

Factors of $\mathcal{M}^{\triangleleft}$ and $\mathcal{M}^{<}$

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FACTORS OF $\mathcal{M}^{\triangleleft}$ AND $\mathcal{M}^{<}$

1. Factors of the *successor* model are essentially *substrings*.
2. Factors of the *precedence* model are essentially *subsequences*.

(Rogers et al. 2013, Rogers and Lambert 2019)

Locally Testable Constraints ≡ PROP($\mathcal{M}^{\triangleleft}$)

Definition: Locally Testable

A constraint is Locally Testable if and only if it can be defined with propositional logic with the successor model where the atoms are factors of the word models.

Examples of Locally Testable Constraints

Which words have models that satisfy φ?

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6. Which of the above are Strictly Local?

Abstract Characterization of LT

Theorem: Local Testability (v1)

A constraint is Locally Testable if and only if there is a number k such that for all $u, v \in \Sigma^*$, if u and v have the same set of k-substrings then either both u, v obey the constraint or neither does.

(McNaughton and Papert 1971, Rogers and Pullum 2011, Rogers et al. 2013, Rogers and Lambert 2019)

Abstract Characterization of LT

Theorem: Local Testability (v2)

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1. Provides a way to decide what is NOT Locally Testable.
2. Provides inference rules for learning.
3. Provides a characterization independent of any formalism.

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In class Exercises

1. Prove that the constraint which says "Words have at least two *a*s" is not LT.

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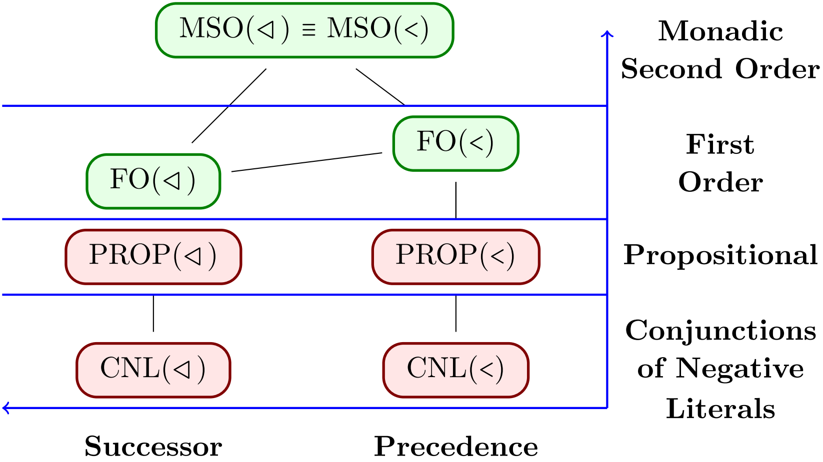
1. Prove that the constraint which says "Words have at least two *a*s" is not LT.
2. You are born on planet Locally 2-Testable, where everyone's DNA has a UG programmed for LT-2 constraints. You observe the word *aab*.
    1. Can you infer that *aaab* is a word in your language?
    2. What about *ab*?

Hierarchies of Constraints



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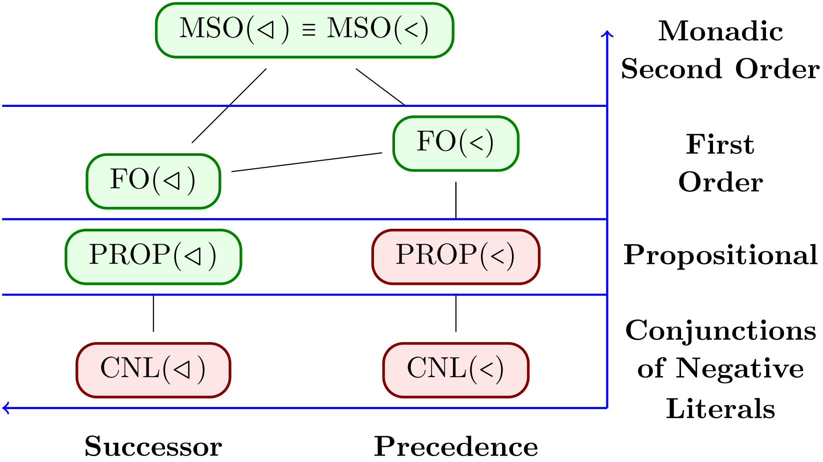

Piecewise Testable Constraints ≡ PROP($\mathcal{M}^<$)

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- The theorem provides a law which simultaneously
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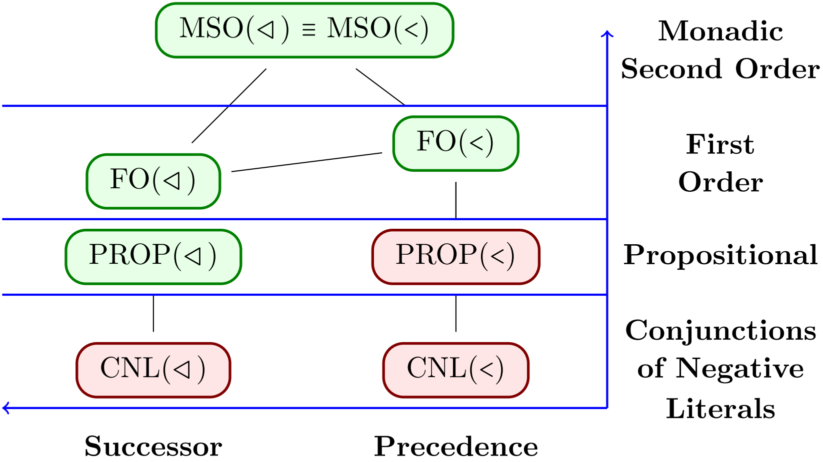

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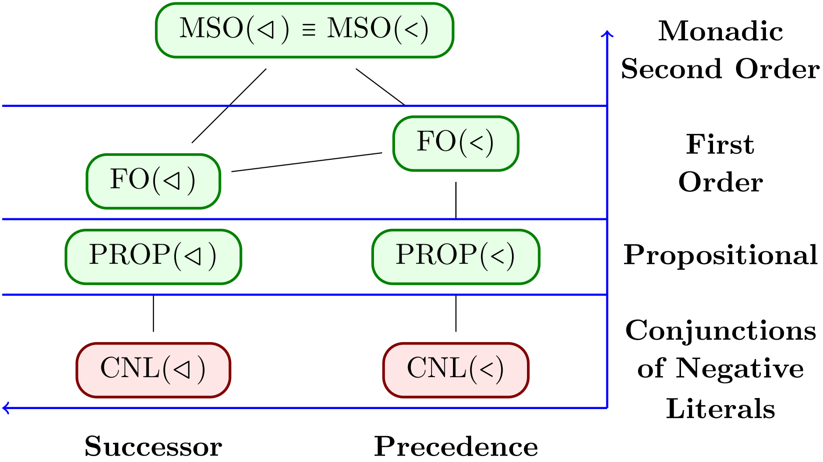

Strictly Local Constraints ≡ CNL($\mathcal{M}^{\lhd}$)

Conjunctions of Negative Literals refer to a fragment of Propositional Logic. They are those sentences with the following form.

$$\varphi \stackrel{\text{def}}{=} \neg \mathrm{P} \wedge \neg \mathrm{Q} \wedge \neg \mathrm{R} \wedge \neg \mathrm{S}$$

- A SL_k grammar can be thought of as a list of **forbidden** sub-strings whose max length is k.
- As a logical language, this is simply the conjunctions of negative literals:
  - where the literals u are strings
  - and $\mathcal{M}_w^{\lhd} \models u$ iff $\mathcal{M}_u^{\lhd}$ is a factor of $\mathcal{M}_w^{\lhd}$.

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Strictly Piecewise Testable Constraints ≡ CNL($\mathcal{M}^<$)

Definition: Stricty Piecewise

A constraint is Strictly Local if and only if it can be defined as the conjunctions of negative literals within a propositional logic with the successor model where the atoms are factors of the word models.

Examples

Examples of Strictly Local constraints for strings

- *aa
- *ab
- *NT
- NoCoda

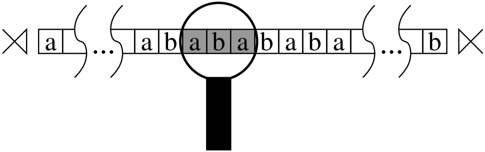
Examples of Non-Strictly Local constraints

- *s...ʃ (Hansson 2001, Rose and Walker 2004, Hansson 2010)
- *#s...ʃ# (Lai 2012, 2015)
- Obligatoriness: Words must contain one primary stress (Hayes 1995, Hyman 2011, inter alia).

Scanning for violations of Strictly Local Constraints

Under the successor word model, **conected sub-structures denote sub-strings** of a certain size.

- We can imagine examining each of the factors of size k, checking to see if it is forbidden or not. The whole structure is well-formed only if each such factor is.



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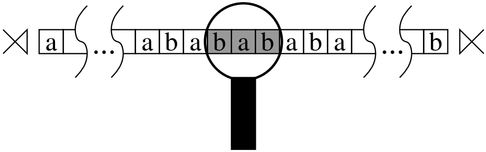

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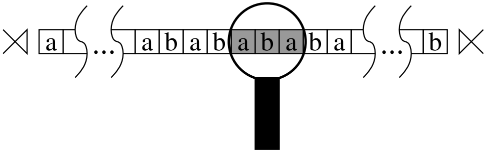

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SL stringsets - Abstract characterization

The theorem below establishes a set-based characterization of SL stringsets independent of any grammar, scanner, or automaton.

Theorem. k-Local Suffix Substitution Closure

For all $L \subseteq \Sigma^*$, $L \in \mathrm{SL}$ iff there exists k such that for all $u_1,\ v_1,\ u_2,\ v_2,\ x \in \Sigma^*$ it is the case that

$$\begin{array}{rl} \text{if} & u_1 x v_1, u_2 x v_2 \in L \text{ and } |x| = k-1 \\ \text{then} & u_1 x v_2 \in L \end{array}$$

Using Suffix Substitution Closure

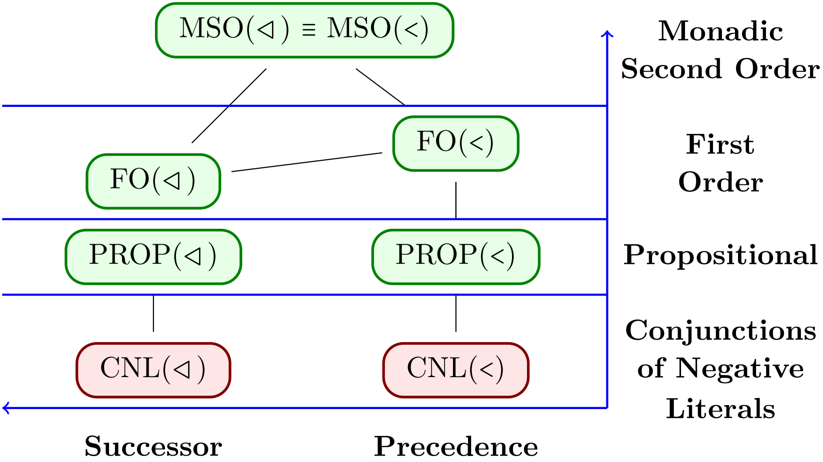
- The theorem provides a law which simultaneously
  - provides a basis for *inference*
  - provides a method for establishing non-SL_k stringsets.

$$
\begin{array}{c|c|cc}
u_1 & \sigma_1 \cdots \sigma_{k-1} & v_1 & \in L \\
u_2 & \sigma_1 \cdots \sigma_{k-1} & v_2 & \in L \\
\hline
u_1 & \sigma_1 \cdots \sigma_{k-1} & v_2 & \in L
\end{array}
$$

In-class exercises

1. Consider a Strictly 2-Local stringset L which contains the words aa and ab. Using Suffix Substitution Closure, explain what other words must be in L.
2. Consider the constraint *s. . . ʃ. Show this is not SL_k for any k.

Hierarchies of Constraints



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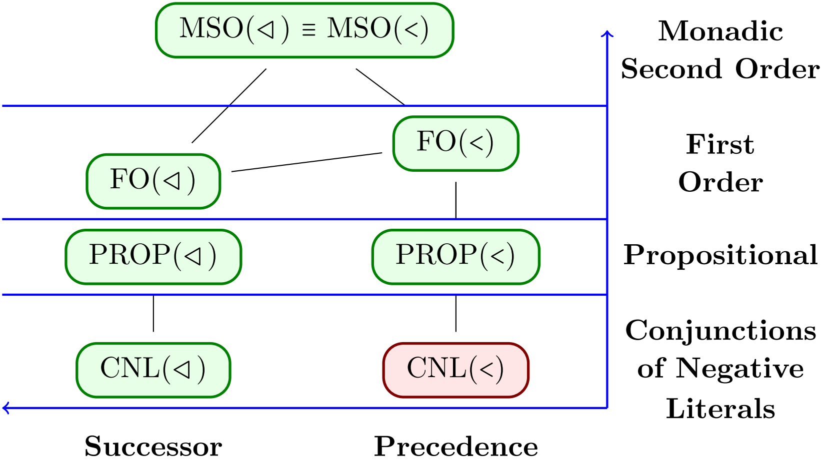

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Abstract Characterization of SP

Subsequence Closure

A constraint is Strictly Piecewise if and only if it is closed under subsequence.

In other words, if a string u obeys the constraint then every one of its subsequences also obeys the constraint.

1. Provides a way to decide what is NOT Strictly Piecewise.
2. Provides inference rules for learning.
3. Provides a characterization independent of any formalism.

(Rogers et al. 2010, 2013)

Filling out Hierarchies of Constraints

- $\boxed{\text{PROP}(\lhd)}$
- $\boxed{\text{PROP}(<)}$
- $\boxed{\text{CNL}(\lhd)}$
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- ...more representations, more logics?

Filling out Hierarchies of Constraints

Tier-based Strictly Local

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Filling out Hierarchies of Constraints

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Multiple Tier-based Strictly Local

Filling out Hierarchies of Constraints

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- This is CNL($\mathcal{M}^{\triangleleft_{T_1}, \triangleleft_{T_2}, \ldots \triangleleft_{T_n}}$)?

Filling out Hierarchies of Constraints

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- This is CNL($\mathcal{M}^{\lhd_T}$).

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Structure-Sensitive Tier-based Strictly Local

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Structure-Sensitive Tier-based Strictly Local

- My guess is this is also CNL with a successor relation that is defined according to the structure-sensitive tier-projection. Somebody ought to look into it!

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Interval-Based Strictly Piecewise

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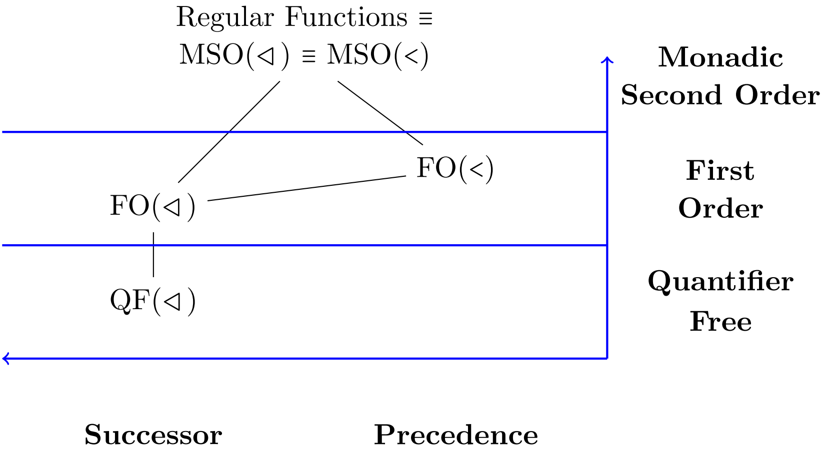
(Heinz et al. 2011, Graf 2017, DeSanto and Graf 2019, Lambert 2020)

Part III

Hierarchies of Transformations

Hierarchies of Transformations



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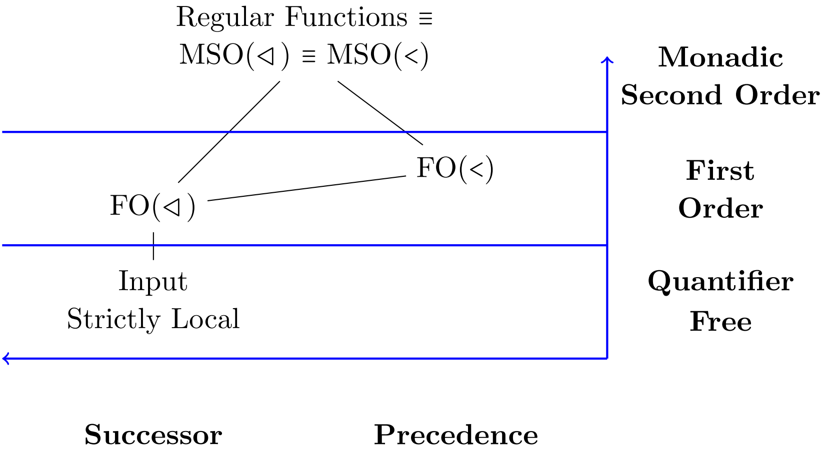
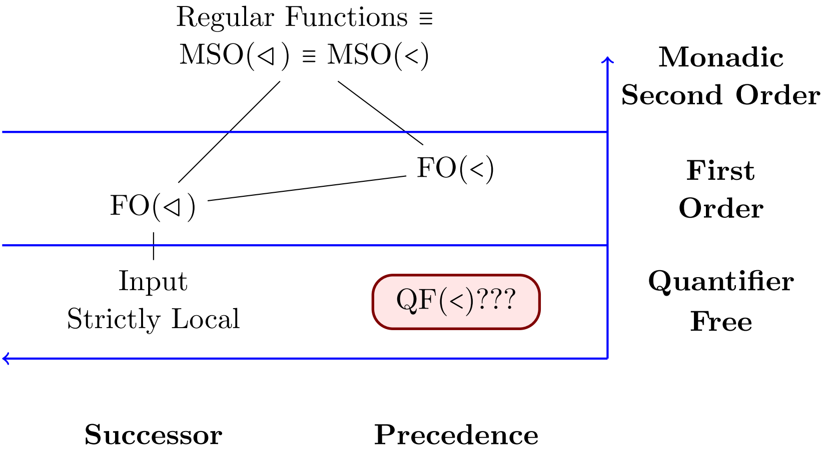


Hierarchies of Transformations



Locality and Quantification

Compare:

1. $P(x) \overset{\text{def}}{=} Q(x) \wedge \exists y[R(y)]$ (First Order Definable)

   Requires scanning whole word for such a y!!

2. $P(x) \overset{\text{def}}{=} Q(x) \wedge R(\texttt{successor}(x))$ (QF Definable)

   Information to decide P is local to x in the input!!

Logical Characterization of ISL Functions

Theorem.

ISL functions correspond exactly to the functions definable with Quantifier-Free logic with successor: QF($\triangleleft$).

(Lindell and Chandlee, LICS 2016)

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Quantifier-free formulas are ones without *any* quantification!

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Instead of

$$C_xC \stackrel{\text{def}}{=} \texttt{cons}(x) \wedge \exists y[x \triangleleft y \wedge \texttt{cons}(y)]$$

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No quantification needed to introduce elements local to x!

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- How can the notion of precedence be employed to describe transformations?

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3. Plenty of room for new ideas here!

Summary

1. Using model signatures with functions instead of relations is a key element to obtaining Quantifier-Free transformations.
2. Characterizing long-distance transformations is a challenging frontier.
3. The constraint hierarchies have two levels below FO: Prop and CNL. The transformation hierarchies only have one level QF, which appears to correspond to CNL. What fragment of FO in the transformation hierarchy corresponds to the Prop level in the constraint hierarchy?

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