1 Semirings

We have seen how we can use logic to describe functions $f : \Sigma^* \to \{\mathtt{true}, \mathtt{false}\}$. Weighted Logics allow one to describe functions with different co-domains, including $\mathbb{N}, [0,1], \Delta^*$ and so on. Crucially, the co-domain is a mathematical object known as a *semiring*. We basically follow the presentation by [DG09].[1]

A semiring is a set S with two binary operations $\oplus, \otimes$, called 'addition/plus' and 'multiplication/times', and with elements 1 and 0 with the following properties satisfied for all $x, y, z \in S$:

(P1)	$x \oplus y, x \otimes y \in S$	(closure under $\oplus$ and $\otimes$)
(P2)	$x \oplus y = y \oplus x$	($\oplus$ is commutative)
(P3)	$0 \oplus x = x \oplus 0 = x$	(0 is the identity for $\oplus$)
(P4)	$1 \otimes x = x \otimes 1 = x$	(1 is the identity for $\otimes$)
(P5)	$0 \otimes x = x \otimes 0 = 0$	(0 is an annihilator for $\otimes$)
(P6)	$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$	($\otimes$ right distributes over $\oplus$)

Below are some examples of semirings.

Name	S	$\oplus$	$\otimes$	0	1	Name	S	$\oplus$	$\otimes$	0	1
Boolean	$\{\mathtt{true}, \mathtt{false}\}$	$\vee$	$\wedge$	`false`	`true`	Viterbi	$[0,1]$	`max`	$\times$	0	1
Natural	$\mathbb{N}$	$+$	$\times$	0	1	Language	$\mathcal{P}(\Sigma^*)$	$\cup$	$\cdot$	$\emptyset$	$\{\lambda\}$

Previously we could understand existential quantification as disjunction over the elements in the domain whereas universal quantification is a conjunction of the elements in the domain. With WMSO, existential quantification combines the elements of the domain with $\oplus$ whereas universal quantification combines them with $\otimes$.

2 Weighted MSO Logic for relational models

Definition 1 (Formulas of WMSO logic)

The base cases. *For all variables $x, y \in \{x_0, x_1, \ldots\}$, $X \in \{X_0, X_1, \ldots\}$, and for all $R \in \mathbb{M}$ the following are formulas of MSO logic.*

(B1)	$\boldsymbol{s}$*, for each* $s \in S$	***(atomic semiring element)***
(B2)	$\boldsymbol{x = y}$	***(equality)***
(B3)	$\boldsymbol{\neg(x = y)}$	***(non-equality)***
(B4)	$\boldsymbol{x \in X}$	***(membership)***
(B5)	$\boldsymbol{\neg(x \in X)}$	***(non-membership)***
(B6)	$\boldsymbol{R(\vec{x})}$*, for each* $R \in \mathbb{M}$	***(positive relational atom)***
(B7)	$\boldsymbol{\neg R(\vec{x})}$*, for each* $R \in \mathbb{M}$	***(negative relational atom)***

It is understood that the $|\vec{x}| = \mathtt{arity}(R)$. So if R is a unary relation, then $\vec{x} = (x)$. If R is a binary relation, then $\vec{x} = (x, y)$, and so on.

[1] An important difference is I have kept equality, which they omit.

The inductive cases. *If φ, ψ are formulas of MSO logic, then so are*

(I1)	$(\boldsymbol{\varphi} \vee \boldsymbol{\psi})$	***(disjunction)***
(I2)	$(\boldsymbol{\varphi} \wedge \boldsymbol{\psi})$	***(conjunction)***
(I3)	$(\exists \boldsymbol{x})[\boldsymbol{\varphi}]$	***(existential quantification for individuals)***
(I4)	$(\exists \boldsymbol{X})[\boldsymbol{\varphi}]$	***(existential quantification for sets of individuals)***
(I5)	$(\forall \boldsymbol{x})[\boldsymbol{\varphi}]$	***(universal quantification for individuals)***
(I6)	$(\forall \boldsymbol{X})[\boldsymbol{\varphi}]$	***(universal quantification for sets of individuals)***

Nothing else is a formula of MSO logic. Note negation only applies to the base cases.

As before, interpreting φ requires a variable assignment function v. We write $[\![\varphi]\!](v, w)$ to express the value in S that φ assigns to $\mathcal{M}_w$ with v. Let D be the domain of $\mathcal{M}_w$. .

Definition 2 (Interpreting formulas of WMSO logic)

The base cases.

(B1)	$[\![\boldsymbol{s}]\!](v, w)$	$\stackrel{def}{=}$	s	
(B2)	$[\![(\boldsymbol{x} = \boldsymbol{y})]\!](v[x \mapsto e_1, y \mapsto e_2], w)$	$\stackrel{def}{=}$	1 *iff* $e_1 = e_2$	, 0 *otherwise*
(B3)	$[\![\neg(\boldsymbol{x} = \boldsymbol{y})]\!](v[x \mapsto e_1, y \mapsto e_2], w)$	$\stackrel{def}{=}$	0 *iff* $e_1 = e_2$	, 1 *otherwise*
(B4)	$[\![\boldsymbol{x} \in \boldsymbol{X}]\!](v[x \mapsto e, X \mapsto E], w)$	$\stackrel{def}{=}$	1 *iff* $e \in E$	, 0 *otherwise*
(B5)	$[\![\neg(\boldsymbol{x} \in \boldsymbol{X})]\!](v[x \mapsto e, X \mapsto E], w)$	$\stackrel{def}{=}$	0 *iff* $e \in E$	, 1 *otherwise*
(B6)	$[\![\boldsymbol{R}(\vec{\boldsymbol{x}})]\!](v[\vec{x} \mapsto \vec{e}], w)$	$\stackrel{def}{=}$	1 *iff* $\mathcal{M}_w \models R(\vec{e})$	, 0 *otherwise*
(B7)	$[\![\neg \boldsymbol{R}(\vec{\boldsymbol{x}})]\!](v[\vec{x} \mapsto \vec{e}], w)$	$\stackrel{def}{=}$	0 *iff* $\mathcal{M}_w \models R(\vec{e})$	, 1 *otherwise*

The inductive cases.

(I1)	$[\![(\boldsymbol{\varphi} \vee \boldsymbol{\psi})]\!](v, w)$	$\stackrel{def}{=}$	$[\![\boldsymbol{\varphi}]\!](v, w) \oplus [\![\boldsymbol{\varphi}]\!](v, w)$
(I2)	$[\![(\boldsymbol{\varphi} \wedge \boldsymbol{\psi})]\!](v, w)$	$\stackrel{def}{=}$	$[\![\boldsymbol{\varphi}]\!](v, w) \otimes [\![\boldsymbol{\varphi}]\!](v, w)$
(I3)	$[\![(\exists \boldsymbol{x})[\boldsymbol{\varphi}]]\!](v, w)$	$\stackrel{def}{=}$	$\bigoplus_{e \in D} [\![\boldsymbol{\varphi}]\!](v[x \mapsto e], w)$
(I4)	$[\![(\exists \boldsymbol{X})[\boldsymbol{\varphi}]]\!](v, w)$	$\stackrel{def}{=}$	$\bigoplus_{E \in D} [\![\boldsymbol{\varphi}]\!](v[X \mapsto E], w)$
(I5)	$[\![(\forall \boldsymbol{x})[\boldsymbol{\varphi}]]\!](v, w)$	$\stackrel{def}{=}$	$\bigotimes_{e \in D} [\![\boldsymbol{\varphi}]\!](v[x \mapsto e], w)$
(I6)	$[\![(\forall \boldsymbol{X})[\boldsymbol{\varphi}]]\!](v, w)$	$\stackrel{def}{=}$	$\bigotimes_{E \in D} [\![\boldsymbol{\varphi}]\!](v[X \mapsto E], w)$

When there is universal quantification over individuals ($\forall x$), the multiplication is done according to the natural order. When there is universal quantification over sets of individuals, an order over the subsets of the domain must be assumed. Since addition is necesarily commutative (unlike multiplication), we do not worry about the order of the computation for existential quantification.

Let Ω be a class of objects (like Σ^*) and let S be a semiring. Let $\mathbb{M}$ denote a model signature for (elements of) Ω. Let φ be a sentence of WMSO($\mathbb{M}$). Then $[\![\varphi]\!]\colon \Omega \to S$.

References

[DG09] Manfred Droste and Paul Gastin. Weighted automata and weighted logics. Monographs in Theoretical Computer Science, chapter 5. Springer, 2009.